ON THE NUMBER OF EUCLIDEAN ORDINARY POINTS FOR LINES IN THE PLANE

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Abstract

Given an arrangement of n not all coincident, not all parallel lines in the (projective or) Euclidean plane we have earlier shown that there must be at least $(5n+6)/39$ Euclidean ordinary points. We improve this result to $n/6$.

1 Sylvester's problem in the Euclidean plane

The classical Theorem of Sylvester and Gallai states that given a set of n not all collinear points in the plane, there must be at least one line which passes through exactly two of the points. The theorem has a corresponding dual statement, namely that any collection of n lines in the projective plane has at least one point where precisely two of the lines intersect. We call such a point an **ordinary** point. The Theorem of Sylvester and Gallai is known to follow from Euler's formula for projective arrangements. See Felsner's excellent treatment of Sylvester problem [3] for details.

Many proofs of the Sylvester-Gallai theorem are known, the first of which was given by Gallai in 1944 [4]. Following the proof of the Sylvester-Gallai theorem, attention turned to giving a lower bound on the number of such ordinary points. In 1958 Kelly and Moser [5] proved that there must be $3n/7$ ordinary lines, and then in 1993 Csima and Sawyer [2] proved that as long as $n \neq 13$, there must be at least $6n/13$ ordinary points.

Recently, Lenchner [6], [7] considered the following variant of the Sylvester problem: In an arrangement of n lines in the Euclidean plane, not all of which all of which are parallel and not of which pass through a common point, must there be a (Euclidean) ordinary point? In [7] it was pointed out that a positive answer to this question does not follow from Euler's formula, but that a bound of $(5n+6)/39$ does follow as a consequence of the Csima-Sawyer bound. In this paper, we improve the $(5n+6)/39$ bound to $n/6$ without using Csima-Sawyer.

2 The $(5n + 6)/39$ Result

Theorem 1 *In an arrangement of n not all collinear, not all coincident lines in the Euclidean plane, there must be at least $(5n + 6)/39$ Euclidean ordinary points.*

Proof. We consider the problem embedded in the real projective plane, where the Csima-Sawyer Theorem [2] says that there must be at least $6n/13$ ordinary points except when $n = 7$. The $n = 7$ case of the Theorem is handled by Lemma 4 from [7] as a simple consequence of the theory of wedges.

If our result were false then more than $\lceil \frac{6n}{13} - \frac{5n+6}{39} \rceil = \lceil \frac{n}{3} - \frac{2}{13} \rceil$ of these ordinary points would have to lie on the line at infinity. In other words there would have to be at least $\lceil \frac{n}{3} - \frac{2}{13} \rceil$ pairs of parallel lines. To this arrangement add the line at infinity. This "kills off" the at least $\lceil \frac{n}{3} - \frac{2}{13} \rceil$ ordinary points and creates at most $\lfloor n - \frac{2n}{3} + \frac{4}{13} \rfloor = \lfloor \frac{n}{3} + \frac{4}{13} \rfloor$ new ordinary points.

By Csima-Sawyer applied to the new arrangement (as long as $n \neq 6$, a case we cover at the end) we have at least $\lceil \frac{6(n+1)}{13} \rceil$

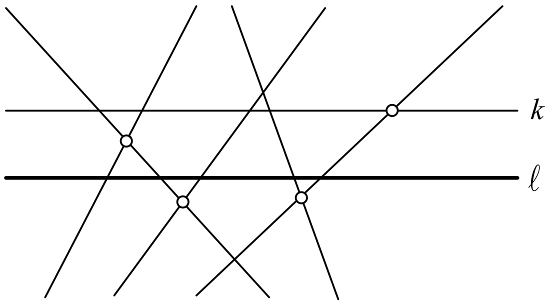

Figure 1: An example of a line ℓ with four ordinary points attached. The lines ℓ and k are drawn to be parallel. The rightmost attached point is attached via an infinite triangle.

ordinary points. But then there must have been at least $\lceil \frac{6(n+1)}{13} - \frac{n}{3} - \frac{4}{13} \rceil = \lceil \frac{5n+6}{39} \rceil$ finite ordinary points earlier, contradicting our initial assumption. The result is thus proved.

If $n = 6$ the theorem claims that there is at least 1 finite ordinary. By the Kelly-Moser Theorem we know that there are at least 3 total ordinary points. If all such points were on the line at infinity the implication would be that we have 3 pairs of parallel lines. Adding a seventh line at infinity would yield a projective arrangement without ordinary points, a clear impossibility. The theorem follows. $\square$

The algebra that allows one to arrive at the number $(5n + 6)/39$ is described in [6].

3 Improving the bound to $n/6$

Definition 2 *Say that an ordinary point p is **attached** to a line l, not containing p, if l together with two lines crossing at p form a (possibly infinite) triangular cell of the arrangement.*

Figure 1 illustrates a line l and its attached points. In this article we focus on finite attached points. Kelly and Moser [5] used the notion of attached points together with a double counting argument to obtain their $3n/7$ bound. Csima and Sawyer [2] added an additional, though highly non-obvious, observation about attached points to those of Kelly and Moser to obtain their $6n/13$ bound. The following simple lemma is used in both papers:

Lemma 3 *In any arrangement of lines, an ordinary point can have at most* 4 *lines counting that point as an attachment.*

Proof. An ordinary point is contained in 2 crossing lines, and hence a vertex of 4 faces; it can therefore be attached to at most 4 lines. □

Our central lemma is the following:

Lemma 4 *Let $\mathcal{A}$ be a Euclidean arrangement of n lines, with not all lines parallel and not all lines passing through a common point. Then if a line $l \in \mathcal{A}$ does not contain an ordinary point, then it must have at least one (Euclidean) ordinary point attached to it.*

Proof. If all the Euclidean vertices are on a single line, then all but that line must be parallel, and all vertices are ordinary. There is thus no line without Euclidean ordinary points.

Thus let $l \in \mathcal{A}$ be a line without Euclidean ordinary points and let x be the closest vertex to l, and rightmost if there are several such vertices. We argue that x must be ordinary. In that case, the triangle defined by l and the two lines through x must be a cell of the arrangement (possibly infinite if one of those lines is parallel to l). Thus x is attached to l. If x is not ordinary, then there are at least three lines through x, let us call them l_1, l_2 and l_3, with l_3 possibly parallel to l, and l_2 intersecting l between l_1 and l_3 (or to the right of l_1 if l_3 is parallel). Then the intersection y of l_2 and l must be non-ordinary, yet any line through it must intersect l_1 or l_3 in a point that is closer to l than x, or to the right of x on l_3 if l_3 is parallel to l, in either case a contradiction. □

Theorem 5 *Let $\mathcal{A}$ be a Euclidean arrangement of n lines, with not all lines parallel and not all lines passing through a common point. Then $\mathcal{A}$ has at least $n/6$ (Euclidean) ordinary points.*

Proof. Let k_i denote the number of lines of $\mathcal{A}$ containing exactly i Euclidean ordinary points, and suppose that there are fewer than $n/6$ Euclidean ordinary points in total. Then we have

$$\sum_{i\geq 1} ik_i < \frac{n}{3} \tag{1}$$

since the sum on the left counts each ordinary point twice.

Also,

$$\sum_{i\geq 0} k_i = n \tag{2}$$

so that

$$\sum_{i\geq 0} k_i > \sum_{i\geq 1} 3ik_i \tag{3}$$

so

$$k_0 > \sum_{i\geq 1} (3i-1)k_i. \tag{4}$$

But also there are at most 4 lines counting a given ordinary point as an attachment (possibly via an infinite triangle), so that if $1+\epsilon_0$ denotes the average number of Euclidean attached points for lines with no Euclidean ordinary points (Lemma 4), and ϵ_i denotes the average number of Euclidean attached points for lines with $i \geq 1$ Euclidean points, we have, for $\epsilon_i \geq 0$, $\forall i \geq 0$,

$$(1+\epsilon_0)k_0 + \sum_{i\geq 1} \epsilon_i k_i \leq 2\sum_{i\geq 1} ik_i \tag{5}$$

i.e.

$$k_0 \leq \sum_{i\geq 1} 2ik_i. \tag{6}$$

But $3i-1 \geq 2i$ for $i \geq 1$, so equations (4) and (6) cannot simultaneously hold and the theorem follows. □

In projective arrangements, since the choice of which is the line at infinity is completely arbitrary, we have the following immediate corollary:

Corollary 6 *Let $\mathcal{A}$ be a projective arrangement of n lines, not all of which pass through a common point. Then there are at least $n/6$ ordinary points off any line which is not part of the arrangement.*

Slightly less obvious is the following:

Corollary 7 *Let $\mathcal{A}$ be a projective arrangement of n lines, no $n-1$ of which pass through a common point. Then there are at least $(n-1)/6$ ordinary points off any line in the arrangement.*

Proof. Let $\ell \in \mathcal{A}$ and consider the arrangement with l removed. By the assumption about no $n-1$ of the lines passing through a common point, we may apply the previous Corollary to conclude that there are at least $(n-1)/6$ ordinary points off of ℓ, points which are ordinary with or without ℓ. $\square$

4 Concluding Remarks

The big open conjecture in the classical Sylvester case, where we consider projective lines and projective ordinary points is that except for $n = 7, 13$ that there must be at least $n/2$ ordinary points. If this conjecture were true, then the methods of section 2 would immediately imply the $n/6$ bound obtained in section 3. Thus the $n/6$ result is in some sense stronger than the Csima-Sawyer bound. A natural question is whether the $n/6$ bound can be used to strengthen Csima and Sawyer's $6n/13$ bound. We think not, since the argument of section 2 involves a double application of the Csima-Sawyer bound, and probably provides a lot of overkill. There are arrangements with exactly $n/2$ ordinary points known for every even n, however if n is odd and

$n \neq 7, 13$ then the worst known cases have $3(n-1)/4$ ordinary points for $n \equiv 1 \pmod 4$ and $3(n-3)/4$ ordinary points for $n \equiv 3 \pmod 4$ (arrangements due to Böröczky; see [1]). In the double application of Csima-Sawyer we are applying the Csima-Sawyer bound for consecutive values of n, hence once for odd n and once for even n.

We thus anticipate that the $n/6$ bound can be improved using new insights and perhaps such an improved bound could in turn lead to a proof of the $n/2$ conjecture.

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